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Mean Field Theory for Coulomb Systems[1]

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We study a classical charge symmetric system with an external charge distribution q in three dimensions in the limit that the plasma parameter $\varepsilon \to$ zero. We prove that if q is scaled appropriately then the correlation functions converge pointwise to those of an ideal gas in the external mean field $\Psi(x)$ where Ψ is given by

$$-\Delta\Psi + 2z\sinh(\beta\Psi) = q$$

This is the mean field equation of Debye and Hückel. The proof uses the sine-Gordon transformation, the Mayer expansion, and a correlation inequality.

KEY WORDS: Debye–Hückel theory; Coulomb systems; mean field theory; sine-Gordon transformation; Mayer expansion.

1. INTRODUCTION

As in [5] we study a classical charge symmetric Coulomb system in three dimensions in the limit that ε tends to zero. ε is the plasma parameter

$$\varepsilon = \beta/l_D \tag{1.1}$$

where β is the inverse temperature, and l_D is the Debye length

$$l_D = (2\beta z)^{-1/2} \tag{1.2}$$

z is the chemical activity. Unlike [5], we include an external charge distribution $q(x)$ in our system. Debye and Hückel [3] used a mean field approximation to study this limit. Our main theorem, Theorem 3.1, says that if we scale the charge distribution $q(x)$ appropriately then in the limit of ε

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tending to zero the correlation functions of our system converge pointwise to those of an ideal gas in an external mean field $\Psi(x)$. Ψ is given by

$$-\Delta\Psi + 2z\sinh(\beta\Psi) = q \tag{1.3}$$

The sine-Gordon transformation expresses the partition function as a functional integral

$$Z = \int d\mu \exp\left\{2z \int_\Lambda dx :\cos[\sqrt{\beta}\,\phi(x)]: + i\int dx\sqrt{\beta}\,\phi(x)\,q(x)\right\}$$

where $d\mu$ is a Gaussian measure whose covariance is essentially $1/|x-y|$. The integrand of this functional integral is the partition function of an ideal gas in an imaginary external potential. Thus the sine-Gordon transformation expresses our system as the average of an ideal gas over different imaginary external potentials. After a scaling argument we will find that each term in the exponential contains a factor of $1/\varepsilon$, as does the inverse of the covariance of $d\mu$. So we should look for stationary points of the functional integral. There is one stationary point, and it is given by

$$\Delta\phi - 2z\sqrt{\beta}\sin(\sqrt{\beta}\,\phi) + i\sqrt{\beta}\,q = 0$$

Comparing this PDE and Eq. (1.3) we see that the dominant term in the functional integral (1.4) is given by $\phi = i\sqrt{\beta}\,\Psi$.

In [5] we studied the same system without an external charge distribution q. This paper should be regarded as complementary to [5] rather than a continuation of [5]. The only parts of [5] that we will use explicitly are the correlation inequalities of Section 5 of [5] and the discussion of the Mayer expansion in Appendix A of [5].

To make the Coulomb system stable we must add a short-range potential, e.g., hard cores, to the Coulomb potential. No such short-range potential appears in the Debye–Hückel theory, so we will let the short-range potential tend to zero as ε tends to zero. We require the same hypotheses on this short-range potential that we did in Section 6 of [5].

The sine-Gordon transformation introduces functional integrals that must be controlled. We do this using a correlation inequality from [5]. This inequality is an extension of an inequality of Fröhlich and Park [4]. Another approach to controlling the functional integrals is to use the cluster expansion of Brydges and Federbush [1]. The main advantage of our approach is its simplicity. Our approach allows several types of boundary conditions while the cluster expansion has only been carried out for Dirichlet boundary conditions. The main advantage of the cluster expansion is that it can handle the non-charge-symmetric case while our approach cannot.

We end our introduction with an outline of the paper. In Section 2 we define the Coulomb system. In Section 3 we state our main theorem and briefly study the PDE (1.3). In Section 4 we use the sine-Gordon transformation and the Mayer expansion to rewrite the partition function and the correlation functions as functional integrals. We then use these results to give a nonrigorous derivation of our main result. We prove our main result in Section 5. The more technical parts of this proof are relegated to Appendices A and B.

2. THE SYSTEM

Our system consists of two species of particles with equal chemical activities z and charges ± 1. We include an external charge distribution in our system. This charge distribution is specified by a real-valued function $q(x)$ on $\mathbb{R}^3$. We will assume $q(x)$ is a C^∞ function with compact support.

The particles interact with each other via the two-body potential

$$V(x, \gamma; y, \delta) = \frac{\gamma\delta}{4\pi\,|x-y|} + V_\varepsilon(x, \gamma; y, \delta) \tag{2.1}$$

where $x, y \in \mathbb{R}^3$ are the positions of the particles and $\gamma, \delta \in \{-1, +1\}$ are their charges. V_ε is a short-range potential which depends on ε and tends to zero as ε tends to zero. For example, V_ε can be the hard-core potential

$$V_\varepsilon(x, \gamma; y, \delta) = \begin{cases} \infty & \text{if } \ |x-y| < 2c_0\varepsilon l_D \\ 0 & \text{otherwise} \end{cases}$$

(c_0 is a constant.) The specific hypotheses that V_ε must satisfy for our theorem are the same as those in [5]. (See the beginning of Section 6 of [5].)

The particles interact with the external charge distribution only through the Coulomb potential $1/4\pi\,|x-y|$. The potential $1/4\pi\,|x-y|$ is the kernel of $-1/\Delta$ where Δ has free boundary conditions. We use free boundary conditions only for convenience. Our theorem is also true for Dirichlet and periodic boundary conditions.

For a volume $\Lambda \subset \mathbb{R}^3$ we denote the grand canonical partition function by $Z_q(\Lambda)$. The correlation function for m particles at $y_1,..., y_m \in \mathbb{R}^3$ with charges $\delta_1,..., \delta_m \in \{-1, +1\}$ is denoted by $\rho^{(m)}_{\Lambda,q}(y_1,..., y_m; \delta_1,..., \delta_m)$. $Z_q(\Lambda)$ and $\rho^{(m)}_{\Lambda,q}$ are defined in the usual way using the potential energy

$$U_n(x_1,..., x_n; \gamma_1,..., \gamma_n) = \sum_{1 \leqslant i<j \leqslant n} V(x_i, \gamma_i; x_j, \gamma_j) + \sum_{i=1}^{n} \gamma_i \int dx \frac{1}{4\pi\,|x_i - x|} q(x) + Q \tag{2.2}$$

where

$$Q = \frac{1}{2} \int dx \int dy\, q(x) \frac{1}{4\pi |x-y|} q(y)$$

For example, see Eqs. (2.4) and (2.6) of [5]. We assume Λ is large enough that it contains the support of q.

We will denote the infinite volume limit of the correlation functions by dropping the subscript Λ. With Dirichlet boundary conditions and a wide class of short range interactions V_ε, these limits were shown to exist by Brydges and Federbush [1]. For arbitrary boundary conditions but special choices of V_ε the existence of these limits was proven by Fröhlich and Park [4]. We assume that some infinite volume limit of our correlation functions exists. The only condition we need on how $\Lambda \to \mathbb{R}^3$ for our result is that bounded sets are eventually contained in Λ.

3. STATEMENT OF RESULT

In the theory of Debye and Hückel [6, pp. 239–242] the mean field $\Psi(x)$ is given by

$$-\Delta \Psi(x) + 2z \sinh[\beta \Psi(x)] = q(x) \tag{3.1}$$

If we work in units with $l_D = 1$, then $\beta = \varepsilon$ and $z = (2\varepsilon)^{-1}$. So in the limit that $\varepsilon \to 0$, Ψ will be given by the linear equation

$$(-\Delta + l_D^{-2})\Psi = q$$

This linearized theory was studied in [5]. To see nonlinear effects we must increase q as $\varepsilon \to 0$.

Without an external charge distribution the densities of the two species of particles are asymptotic to z as $\varepsilon \to 0$ [Theorem 3.3 of 5]. So it is natural to multiply q by z. For later convenience we will multiply q by $2z$. Then (3.1) becomes

$$-\Delta \Psi(x) + 2z \sinh[\beta \Psi(x)] = 2zq(x) \tag{3.2}$$

Let

$$\begin{aligned} \psi(x) &= \beta \Psi(x l_D) \\ \tilde{q}(x) &= q(x l_D) \end{aligned} \tag{3.3}$$

Then (3.2) becomes

$$-\Delta \psi + \sinh \psi = \tilde{q} \tag{3.4}$$

Our main result is the following theorem.

Theorem 3.1. Let $\tilde{q}$ be C^∞ with compact support. Let ψ be given by Eq. (3.4). Let

$$q(x) = \tilde{q}(x/l_D) \tag{3.5}$$

Let $y_1, \ldots, y_m$ be distinct points in $\mathbb{R}^3$ and $\delta_1, \ldots, \delta_m \in \{-1, +1\}$. Then for any choice of potentials V_ε satisfying hypotheses (H1) through (H5) of [5] (see p. 279 of [5])

$$\rho_{2zq}^{(m)}(y_1 l_D, \ldots, y_m l_D; \delta_1, \ldots, \delta_m) \sim z^m \exp\left[-\sum_{j=1}^{m} \delta_j \psi(y_j)\right]$$

as $\varepsilon \to 0$ in the sense that

$$\lim_{\varepsilon \to 0} z^{-m} \rho_{2zq}^{(m)}(y_1 l_D, \ldots) = \exp\left[-\sum_{j=1}^{m} \delta_j \psi(y_j)\right] \tag{3.6}$$

We remind the reader that $\rho_{2zq}^{(m)}$ is the infinite volume limit of the correlation function. So in our theorem the infinite volume limit is taken before the $\varepsilon \to 0$ limit.

Theorem 3.1 assumes that the PDE (3.4) has a solution for $\tilde{q} \in C_0^\infty$. Standard techniques in the theory of PDEs can be used to prove the following lemma.

Lemma 3.2. If $\tilde{q} \in C_0^\infty(\mathbb{R}^3)$, then Eq. (3.4) has a unique, bounded C^∞ solution ψ which is in all the Sobolev spaces, i.e.,

$$\int d^3x\, \psi(x)(-\Delta + 1)^m\, \psi(x) < \infty \qquad \text{for} \quad m = 0, 1, 2, \ldots \tag{3.7}$$

Moreover,

$$|\psi(x)| \leqslant (-\Delta + 1)^{-1} |\tilde{q}|\,(x) \qquad \text{for all} \quad x \in \mathbb{R}^3 \tag{3.8}$$

We sketch a few ideas that can be used to prove Lemma 3.2. Formally, ψ should be the minimum of

$$E(\phi) = \int_{\mathbb{R}^3} \left[\frac{1}{2}\phi(-\Delta\phi) + \cosh\phi - 1 - \phi\tilde{q}\right]$$

$E(\phi)$ is a convex, lower semicontinuous functional on

$$\left\{\phi \in \mathscr{H}_{+1} : \int (\cosh\phi - 1) < \infty\right\}$$

$\mathscr{H}_{+1}$ is the Sobolev space of L^2 functions whose distributional first derivatives are L^2 functions. By a standard theorem [9] there is a unique ψ which minimizes $E(\phi)$.

We would like to use the usual bootstrap argument to show that since $\tilde{q}$ is in all the Sobolev spaces then so is ψ. The exponential growth of the sinh function and its derivatives could cause problems in this argument. However, by the maximum principle and the fact that $\sinh(x) - x \geqslant 0$ for all $x \geqslant 0$, we have the *a priori* bound

$$\sup_x |\psi(x)| \leqslant \sup_x |\tilde{q}(x)| \tag{3.9}$$

So the exponential growth of $\sinh(x)$ is irrelevant.

One way to prove inequality (3.8) is as follows. Let

$$V(x) = \frac{\sinh[\psi(x)]}{\psi(x)}$$

Then the PDE can be written

$$-\Delta\psi + V\psi = \tilde{q}$$

Note that $V(x) \geqslant 1$ for all x. So the Feynman–Kac formula implies that the kernel of $(-\Delta + V)^{-1}$ is pointwise bounded by the kernel of $(-\Delta + 1)^{-1}$. The bound (3.8) follows.

For later reference we summarize some consequences of Lemma 3.2. By inequality (3.8), $\psi(x)$ decays exponentially with the distance from x to the support of $\tilde{q}$. Since $\psi \in L^\infty$ this implies $\psi \in L^1$. ψ being in $L^1 \cap L^\infty$ implies

$$\sinh \psi \in L^1 \cap L^\infty \tag{3.10}$$

Using the PDE one computes

$$\Delta\Delta\psi = \cosh\psi[\sinh\psi - \tilde{q}] + \sinh(\psi)(\nabla\psi) \cdot (\nabla\psi) - \Delta\tilde{q}$$

Lemma 3.2 and the above results show this is in L^1. So

$$\Delta\Delta\psi \in L^1 \tag{3.11}$$

4. THE SINE-GORDON TRANSFORMATION

As in [5] we follow Brydges and Federbush [1] and apply the sine-Gordon transformation to the long-range part of the interaction and use a Mayer expansion for the short-range part. We refer the reader to Appendix A of [5] for the details of the Mayer series. That appendix uses Brydges and Federbush's development of the Mayer series [2].

We take the short- and long-range parts of the Coulomb interaction to be

$$\mathscr{V}_L(x) = \frac{1 - \exp(-|x|/\mu l_D)}{4\pi |x|}$$
$$\mathscr{V}_Y(x) = \frac{\exp(-|x|/\mu l_D)}{4\pi |x|} \tag{4.1}$$

Let

$$V_L(x, \gamma; y, \delta) = \gamma\delta\, \mathscr{V}_L(x-y)$$
$$V_Y(x, \gamma; y, \delta) = \gamma\delta\, \mathscr{V}_Y(x-y) \tag{4.2}$$

So

$$V = V_L + V_Y + V_\varepsilon$$

We define the total short-range interaction to be

$$V_S = V_Y + V_\varepsilon \tag{4.3}$$

As a guide to our notation we offer the following. L, Y, and S stand for long, Yukawa, and short, respectively. We have labeled the short-range part of the Coulomb interaction as V_Y rather than V_S since there is another short-range potential, namely, V_ε. So V_S is used for the sum of these two short-range potentials. We will use the subscripts L, Y, and S in this way with other quantities.

In Section 2 we defined $Z_q(\Lambda)$ and $\rho^{(m)}_{\Lambda,q}$. Our theorem concerns $\rho^{(m)}_{\Lambda,2zq}$, so in this section we will carry out the sine-Gordon transformation for $Z_{2zq}(\Lambda)$ and $\rho^{(m)}_{\Lambda,2zq}$. It is convenient to work in units with $l_D = 1$. (See p. 280 of [5] for the details of how this is done.) This amounts to setting $l_D = 1$, $\beta = \varepsilon$, $2z = \varepsilon^{-1}$ and replacing q by $\tilde{q}$. We should also replace Λ by $l_D^{-1}\Lambda$. To avoid this factor of l_D^{-1} we will write down expressions for $Z_{2zq}(l_D\Lambda)$ and $\rho^{(m)}_{l_D\Lambda,2zq}$. We can do this since we take the infinite volume limit of $\rho^{(m)}_{l_D\Lambda,2zq}$ before we let $\varepsilon \to 0$.

We split the field generated by q and the self-energy of q into short- and long-range parts:

$$A_Y(x, \gamma) = \gamma \int dy\, \mathscr{V}_Y(x-y) \frac{1}{\varepsilon} \tilde{q}(y) \tag{4.4}$$

$$Q_Y = \frac{1}{2} \int dx \int dy \frac{1}{\varepsilon} \tilde{q}(x)\, \mathscr{V}_Y(x-y) \frac{1}{\varepsilon} \tilde{q}(y) \tag{4.5}$$

$A_L(x, \gamma)$ and Q_L are defined similarly. We will use i to stand for x_i, γ_i. For example, $V_L(i, j)$ is $V_L(x_i, \gamma_i; x_j, \gamma_j)$.

The function $C(x, y) = \mathscr{V}_L(x - y)$ is the kernel of the positive operator

$$C = \frac{1}{-\Delta} - \frac{1}{-\Delta + (\mu l_D)^{-2}}$$

Hence there exists a Gaussian process with covariance $C(x, y)$, i.e., there exists a probability measure $d\mu$ and a Gaussian random variable $\phi(x)$ for each $x \in \mathbb{R}^3$ such that $\int d\mu\, \phi(x)\, \phi(y) = C(x, y)$. See pp. 16–17 of [7].

In integrations with respect to Lebesgue measure we will often suppress the dx and the (x) in $\phi(x)$. For example,

$$\int \phi\tilde{q} = \int dx\, \phi(x)\, \tilde{q}(x)$$

We will follow the usual convention of using ϕ to denote a point in the measure space on which $d\mu$ is defined. So $F(\phi)$ is a function on this measure space, and $\sup_\phi |F(\phi)|$ is the sup of $|F(\phi)|$ over the measure space.

The sine-Gordon transformation says

$$Z_{2zq}(l_D \Lambda) = \int d\mu\, Z(\phi)$$

with

$$Z(\phi) = \exp\left[-\varepsilon Q_Y + \frac{i}{\varepsilon}\int \sqrt{\varepsilon}\, \phi\tilde{q}\right] \sum_{n=0}^{\infty} \frac{\hat{z}^n}{n!} \sum_{\gamma_1, \ldots, \gamma_n} \int_\Lambda d^n x$$
$$\times \exp\left\{-\varepsilon\left[\sum_{i<j} V_S(i, j) + \sum_i A_Y(i) - \frac{i}{\sqrt{\varepsilon}} \sum_i \gamma_i \phi(x_i)\right]\right\} \tag{4.6}$$

where $\hat{z} = z \exp(\varepsilon/8\pi\mu)$, $\int_\Lambda d^n x = \int_\Lambda dx_1 \cdots \int_\Lambda dx_n$ and each γ_i is summed over ± 1. The expression beginning with $\sum_{n=0}^{\infty}$ is a partition function with a convergent Mayer series. So

$$Z(\phi) = \exp\left[-\varepsilon Q_Y + \frac{i}{\varepsilon}\int \sqrt{\varepsilon}\, \phi\tilde{q} + \sum_{n=1}^{\infty} K_n(\phi)\right] \tag{4.7}$$

$K_n(\phi)$ is given by Eq. (A.1) of [5] with the one-body interaction being

$$V_1(i) = V_1(x_i, \gamma_i) = A_Y(x_i, \gamma_i) - \frac{i}{\sqrt{\varepsilon}} \gamma_i \phi(x_i) \tag{4.8}$$

We will check the convergence of this Mayer series in Section 5.

For the correlation functions we let

$$\bar{Q}_S = Q_Y + \sum_{1 \leqslant i < j \leqslant m} V_S(y_i, \delta_i; y_j, \delta_j) + \sum_{j=1}^{m} A_Y(y_j, \delta_j) \tag{4.9}$$

$$\bar{A}_S(x, \gamma) = A_Y(x, \gamma) + \sum_{j=1}^{m} V_S(x, \gamma; y_j, \delta_j) \tag{4.10}$$

$\bar{Q}_L$ and $\bar{A}_L$ are defined by the same equations with V_S replaced by V_L, A_Y by A_L, and Q_Y by Q_L. We have

$$\begin{aligned} &\rho^{(m)}_{l_D\Lambda, 2zq}(y_1 l_D, ..., y_m l_D; \delta_1, ..., \delta_m) \\ &\quad = z^m Z_{2zq}(l_D \Lambda)^{-1} \int d\mu \prod_{j=1}^{m} :\exp[i \sqrt{\varepsilon}\, \delta_j \phi(y_j)]: \bar{Z}(\phi) \end{aligned} \tag{4.11}$$

where $\bar{Z}(\phi)$ is given by the equation for $Z(\phi)$, Eq. (4.6), with Q_Y replaced by $\bar{Q}_S$ and A_Y by $\bar{A}_S$. The : : denotes normal ordering. (See p. 275 of [5] or pp. 9–11 of [8].) Note that

$$z{:}\exp[i \sqrt{\varepsilon}\, \delta_j \phi(y_j)]{:} = \hat{z} \exp[i \sqrt{\varepsilon}\, \delta_j \phi(y_j)]$$

Again, we have a convergent Mayer expansion:

$$\bar{Z}(\phi) = \exp\left[-\varepsilon \bar{Q}_S + \frac{i}{\varepsilon} \int \sqrt{\varepsilon}\, \phi \tilde{q} + \sum_{n=1}^{\infty} \bar{K}_n(\phi)\right] \tag{4.12}$$

$\bar{K}_n(\phi)$ is given by Eq. (A.1) of [5] with

$$V_1(i) = V_1(x_i, \gamma_i) = \bar{A}_S(x_i, \gamma_i) - \frac{i}{\sqrt{\varepsilon}} \gamma_i \phi(x_i) \tag{4.13}$$

We can use the results of the sine-Gordon transformation to give a nonrigorous derivation of our main result. This derivation serves as an introduction to the proof in the next section.

If we let $\mu \to 0$ as $\varepsilon \to 0$, then $V_S \to 0$. Setting $V_S = 0$,

$$\begin{aligned} &K_n(\phi) = \bar{K}_n(\phi) = 0 \qquad \text{for} \quad n \geqslant 2 \\ &K_1(\phi) = \bar{K}_1(\phi) = \frac{1}{\varepsilon} \int_\Lambda dx\, {:}\cos[\sqrt{\varepsilon}\, \phi(x)]{:} \end{aligned} \tag{4.14}$$

The measure $d\mu$ is formally given by

$$d\mu(\phi) = \mathscr{D}\phi \exp\left[-\frac{1}{2} \int dx\, \phi(x)(-\Delta\phi)(x)\right]$$

with

$$\mathscr{D}\phi = N^{-1} \prod_{x \in \mathbb{R}^3} d\phi(x)$$

where $d\phi(x)$ is Lebesgue measure with respect to the variable $\phi(x)$. So if we ignore the normal ordering

$$Z_{2zq} = \int \mathscr{D}\phi \exp\left[-\frac{1}{2}\int \phi(-\Delta)\phi + \frac{1}{\varepsilon}\int_{\Lambda} \cos(\sqrt{\varepsilon}\,\phi) + \frac{i}{\varepsilon}\int \sqrt{\varepsilon}\,\phi\tilde{q}\right]$$

We change variables $\phi \to \varepsilon^{-1/2}\phi$. Then

$$Z_{2zq} = \int \mathscr{D}\phi \exp\left[\frac{1}{\varepsilon} S(\phi)\right] \tag{4.15}$$

with

$$S(\phi) = -\frac{1}{2}\int \phi(-\Delta)\phi + \int_{\Lambda} \cos\phi + i\int \phi\tilde{q} \tag{4.16}$$

As $\varepsilon \to 0$ the dominant contribution to this integral should come from the stationary point of $S(\phi)$. It is given by the equation

$$\frac{\delta S}{\delta\phi(x)} = 0$$

or

$$(\Delta\phi)(x) - \sin[\phi(x)] + i\tilde{q}(x) = 0 \tag{4.17}$$

(for $x \in \Lambda$). Letting $\phi = i\psi$, the equation becomes

$$-\Delta\psi + \sinh\psi = \tilde{q}$$

which is Eq. (3.4), the mean field equation of Debye and Hückel.

The correlation functions are given by

$$\rho_{2zq}^{(m)}(y_1 l_D, \ldots; \delta_1, \ldots)$$
$$= z^m Z_{2zq}^{-1} \int \mathscr{D}\phi \prod_j \exp[i\delta_j \phi(y_j)] \exp\left[\frac{1}{\varepsilon} S(\phi)\right] \tag{4.18}$$

The stationary point of this functional integral is the same as that of Z_{2zq}. Evaluating $\prod_j \exp[i\delta_j\phi(y_j)]$ at the stationary point $\phi = i\psi$, we have our main result

$$\rho_{2zq}^{(m)}(y_1 l_D, \ldots) \sim z^m \prod_j \exp[-\delta_j \psi(y_j)] \tag{4.19}$$

5. PROOF

This section is devoted to the proof of Theorem 3.1. We will make frequent use of Section 5 and Appendix A of [5].

We will use the $\mathcal{O}$ and o notation in the following way. A quantity F is $\mathcal{O}(\varepsilon^p)$ if $|F| \leqslant c\varepsilon^p$ for some constant c. c must be independent of Λ. In fact, c can only depend on the charge distribution $\tilde{q}$, the integer m (the theorem concerns the mth correlation function $\rho^{(m)}$), the points $y_1, ..., y_m$, and the constants c_1, δ_1, c_2, δ_2, B, c_3 in hypotheses (H1) through (H5) on V_ε. F is $o(\varepsilon^p)$ if c can be replaced by $f(\varepsilon)$ where f is a positive function with $\lim_{t \to 0^+} f(t) = 0$. f can only depend on the same quantities that c can depend on. As a special case of this notation, a quantity is $\mathcal{O}(1)$ if it is a bounded function of everything except possibly $\tilde{q}$, m, $y_1, ..., y_m$, c_1, δ_1, c_2, δ_2, B, and c_3. If $F = F(\phi)$ is a function on the probability space of our Gaussian process, then $F = \mathcal{O}(\varepsilon^p)$ or $o(\varepsilon^p)$ implies uniformity in ϕ as well. So $F(\phi) = \mathcal{O}(\varepsilon^p)$ means $\sup_\phi |F(\phi)| = \mathcal{O}(\varepsilon^p)$.

We will use $o(\Lambda)$ to denote a quantity that $\to 0$ as $\Lambda \to \mathbb{R}^3$. In general this quantity will depend on ε and need not $\to 0$ uniformly in ε as $\Lambda \to \mathbb{R}^3$.

Throughout the proof we will work with the finite volume correlation function $\rho^{(m)}_{2zq, l_D\Lambda}(y_1 l_D, ...)$. We will show

$$z^{-m} \rho^{(m)}_{2zq, l_D\Lambda}(y_1 l_D, ...) - \exp\left[-\sum_{j=1}^{m} \delta_j \psi(y_j)\right] = o(1) + o(\Lambda)$$

Letting $\Lambda \to \mathbb{R}^3$, this proves the theorem.

Step 1 (Complex Translation). The translation $\phi \to \phi + i\varepsilon^{-1/2}\psi$ yields

$$Z_{2zq}(l_D\Lambda) = \int d\mu \exp[S(\phi) + E] \tag{5.1}$$

with

$$S(\phi) = -\frac{i}{\varepsilon}\int \sqrt{\varepsilon}\, \phi C^{-1}\psi + \frac{i}{\varepsilon}\int \sqrt{\varepsilon}\, \phi\tilde{q} + \sum_{n=1}^{\infty} K_n(\phi + i\varepsilon^{-1/2}\psi)$$

$$E = \frac{1}{2\varepsilon}\int \psi C^{-1}\psi - \varepsilon Q_Y - \frac{1}{\varepsilon}\int \psi\tilde{q}$$

and

$$\begin{aligned} &z^{-m}\rho^{(m)}_{2zq, l_D\Lambda}(y_1 l_D, ...) \\ &\quad = Z_{2zq}(l_D\Lambda)^{-1} \exp\left[-\sum_{j=1}^{m} \delta_j \psi(y_j)\right] \\ &\quad \times \int d\mu \prod_{j=1}^{m} :\exp[i\sqrt{\varepsilon}\, \delta_j \phi(y_j)]: \exp[S(\phi) + S'(\phi) + E + E'] \end{aligned} \tag{5.2}$$

with

$$S'(\phi) = \sum_{n=1}^{\infty} [\bar{K}_n(\phi + i\varepsilon^{-1/2}\psi) - K_n(\phi + i\varepsilon^{-1/2}\psi)] \tag{5.3}$$

$$E' = -\varepsilon\bar{Q}_S + \varepsilon Q_Y \tag{5.4}$$

We split $S(\phi)$ up as

$$S(\phi) = R(\phi) + iI(\phi) + W(\phi)$$

with

$$R(\phi) = \sum_{n=1}^{\infty} \operatorname{Re}[K_n(\phi + i\varepsilon^{-1/2}\psi) - K_n^0(\phi)] - \frac{1}{\varepsilon}\int_\Lambda :\cos(\sqrt{\varepsilon}\,\phi):[\cosh(\psi) - 1]$$

$$I(\phi) = -\frac{1}{\varepsilon}\int \sqrt{\varepsilon}\,\phi C^{-1}\psi + \frac{1}{\varepsilon}\int \sqrt{\varepsilon}\,\phi\tilde{q} + \sum_{n=1}^{\infty} \operatorname{Im}[K_n(\phi + i\varepsilon^{-1/2}\psi) - K_n^0(\phi)]$$

$$W(\phi) = \sum_{n=1}^{\infty} K_n^0(\phi) + \frac{1}{\varepsilon}\int_\Lambda :\cos(\sqrt{\varepsilon}\,\phi):[\cosh(\psi) - 1] \tag{5.5}$$

$K_n^0(\phi)$ is $K_n(\phi)$ with $\tilde{q}$ set equal to zero. So the one-body potential in $K_n^0(\phi)$ is

$$V_1(x, \gamma) = -\frac{i}{\sqrt{\varepsilon}}\gamma\phi(x) \tag{5.6}$$

As we will see later, $R + iI$ is the part of $S(\phi)$ which is bounded as $\Lambda \to \mathbb{R}^3$ and which $\to 0$ as $\varepsilon \to 0$. The charge symmetry of our system implies that $K_n^0(\phi)$ is real, so R, I, and W are real.

For a function $F(\phi)$ on the measure space of $d\mu$ we let

$$\langle F(\phi)\rangle = \frac{\int d\mu\, F(\phi)\exp[W(\phi)]}{\int d\mu\, \exp[W(\phi)]} \tag{5.7}$$

Then

$$z^{-m}\rho^{(m)}_{2zq,l_D\Lambda}(y_1 l_D, \ldots) = \exp\left[-\sum_{j=1}^{m}\delta_j\psi(y_j) + E'\right] \times \frac{\langle\prod_{j=1}^{m} :\exp[i\sqrt{\varepsilon}\,\delta_j\phi(y_j)]: \exp[R(\phi) + iI(\phi) + S'(\phi)]\rangle}{\langle\exp[R(\phi) + iI(\phi)]\rangle}$$

So the proof is reduced to showing

$$E' = o(1) \tag{5.8}$$

$$\langle \exp[R(\phi) + iI(\phi)] \rangle = 1 + o(1) + o(\Lambda) \tag{5.9}$$

$$\left\langle \prod_{j=1}^{m} :\exp[i\sqrt{\varepsilon}\, \delta_j \phi(y_j)]: \exp[R(\phi) + iI(\phi) + S'(\phi)] \right\rangle$$
$$= 1 + o(1) + o(\Lambda) \tag{5.10}$$

Step 2 (Preliminaries). We let

$$\mu = \varepsilon^{1/2+\delta} \qquad \text{with} \quad 0 < \delta < 1/6 \tag{5.11}$$

Then the various short-range forces all go to zero as ε goes to zero. In particular, from Eq. (4.4) and

$$\int dx\, \mathscr{V}_Y(x) = \mu^2$$

we have

$$\|A_Y\|_\infty \leqslant \frac{1}{\varepsilon} \mu^2 \|\tilde{q}\|_\infty = \varepsilon^{2\delta} \|\tilde{q}\|_\infty \tag{5.12}$$

and

$$\|A_Y\|_1 \leqslant \frac{1}{\varepsilon} \mu^2 \|\tilde{q}\|_1 = \varepsilon^{2\delta} \|\tilde{q}\|_1 \tag{5.13}$$

Also,

$$\|V_Y\|_1 = 2\mu^2 \tag{5.14}$$

where $\| \;\|_1$ for a two-body potential is defined by Eq. (6.3) of [5].

Standard techniques [2] show that

$$|\bar{K}_n(\phi + i\varepsilon^{-1/2}\psi)| = \mathscr{O}(\varepsilon^{-1})\, r^{n-1} |\Lambda|$$

with

$$r = \mathscr{O}(1)[\|V_Y\|_1 + \|V_\varepsilon^0\|_1 + \frac{1}{\varepsilon} \|V_\varepsilon^r\|_r]$$

$\|V_\varepsilon^0\|_1$ and $\|V_\varepsilon^r\|_r$ are defined by Eq. (6.3) of [5]. Using Eq. (5.14), hypotheses (H1) and (H2), and our definition of μ, we see

$$r = o(\varepsilon) \tag{5.15}$$

The same bound holds for $K_n(\phi + i\varepsilon^{-1/2}\psi)$ and $K_n^0(\phi)$. So for sufficiently small ε, r will be less than 1, which implies that all our Mayer series converge.

We claim $\langle\ \rangle$ is a sine-Gordon probability measure. (See Definition 5.1 of [5].) To see this we think of $\exp[W(\phi)]$ as a partition function of a system with four species in an imaginary external field $-i\varepsilon^{-1/2}\phi$. Two of the species come from the $\sum_{n=1}^{\infty} K_n^0(\phi)$ term and the other two from the $(1/\varepsilon)\int_\Lambda :\cos(\sqrt{\varepsilon}\,\phi):[\cosh(\psi)-1]$ term. By undoing the Mayer expansion $\exp[W(\phi)]$ can be written as

$$\int d\nu(\rho)\exp[i\phi(\rho)]$$

for some positive measure $d\nu$. (See step 2 of the proof in Appendix A for details.) Thus $\langle\ \rangle$ is a sine-Gordon measure.

Step 3 [Proof of Eq. (5.8)]. From Eqs. (4.9) and (5.4)

$$E' = -\varepsilon\left[\sum_{1\leqslant i<j\leqslant m} V_S(y_i,\delta_i; y_j,\delta_j) + \sum_{j=1}^{m} A_Y(y_j,\delta_j)\right] \tag{5.16}$$

It follows easily from Eq. (5.12), hypothesis (H5), and the choice of μ that

$$E' = \sigma(\varepsilon) \tag{5.17}$$

This proves Eq. (5.8).

Step 4 [Bounding $R(\phi)$]. We separate out the $n=1$ term in the Mayer series in $R(\phi)$. Note that

$$\begin{aligned} K_1(\phi + i\varepsilon^{-1/2}\psi) &= \frac{1}{2\varepsilon}\sum_\gamma \int_\Lambda dx :\exp[i\sqrt{\varepsilon}\,\gamma\phi(x)]: \exp[-\gamma\psi - \varepsilon A_Y(x,\gamma)] \\ K_1^0(\phi) &= \frac{1}{\varepsilon}\int_\Lambda dx :\cos(\sqrt{\varepsilon}\,\phi): \end{aligned} \tag{5.18}$$

So

$$R(\phi) = R_1(\phi) + R_2(\phi)$$

with

$$\begin{aligned} R_1(\phi) &= \sum_{n=2}^{\infty} \operatorname{Re}[K_n(\phi + i\varepsilon^{-1/2}\psi) - K_n^0(\phi)] \\ R_2(\phi) &= \frac{1}{2\varepsilon}\sum_\gamma \int_\Lambda dx :\cos(\sqrt{\varepsilon}\,\phi): e^{-\gamma\psi}[e^{-\varepsilon A_Y(x,\gamma)} - 1] \end{aligned} \tag{5.19}$$

We will show

$$R_i(\phi) = o(1) \qquad \text{for} \quad i = 1, 2 \tag{5.20}$$

We bound $R_1(\phi)$ by bounding $|K_n(\phi + i\varepsilon^{-1/2}\psi) - K_n^0(\phi)|$. This is a difference of two Mayer series, so we can use Lemma A.1 of [5]. We apply the lemma with

$$\begin{aligned} V_1(x, \gamma) &= -\frac{i}{\sqrt{\varepsilon}}\gamma\phi(x) \\ \bar{V}_1(x, \gamma) &= \frac{1}{\varepsilon}\gamma\psi(x) + A_Y(x, \gamma) \end{aligned} \tag{5.21}$$

Then

$$\|\bar{V}_1\|_- = \mathcal{O}(\varepsilon^{-1})$$

($\|\bar{V}_1\|_-$ is the sup norm of the negative part of the real part of $\bar{V}_1$.) Using $\psi \in L^1 \cap L^\infty$ and Eqs. (5.12) and (5.13)

$$\sum_\gamma \int dx\, |\exp[-\varepsilon \bar{V}_1(x, \gamma)] - 1| = \mathcal{O}(1) \tag{5.22}$$

So the lemma says

$$|K_n(\phi + i\varepsilon^{-1/2}\psi) - K_n^0(\phi)| = \mathcal{O}(\varepsilon^{-1})(r')^{n-1}$$

Since $r = o(\varepsilon)$ [see Eq. (5.15)], $r' = o(\varepsilon)$. Hence

$$\sum_{n=2}^{\infty} |K_n(\phi + i\varepsilon^{-1/2}\psi) - K_n^0(\phi)| = o(1) \tag{5.23}$$

This proves Eq. (5.20) for $i = 1$.

Equations (5.12) and (5.13) imply

$$\int_\Lambda dx\, |\exp[-\varepsilon A_Y(x, \gamma)] - 1| = o(\varepsilon) \tag{5.24}$$

Along with

$$|{:}\cos(\sqrt{\varepsilon}\,\phi){:}| \leqslant \exp\left(\frac{\varepsilon}{8\pi\mu}\right)$$

this implies Eq. (5.20) for $i = 2$.

Step 5 [Bounding $I(\phi)$]. Using $C^{-1} = -\Delta + \mu^2\Delta\Delta$ and the fact that ψ solves the PDE (3.4) we have

$$I(\phi) = \frac{1}{\varepsilon}\int \sqrt{\varepsilon}\,\phi[\sinh(\psi) - \mu^2\Delta\Delta\psi] + \sum_{n=1}^{\infty} \operatorname{Im}[K_n(\phi + i\varepsilon^{-1/2}\psi) - K_n^0(\phi)] \tag{5.25}$$

As with $R(\phi)$ we separate out the $n = 1$ term:

$$\begin{aligned} I(\phi) &= \sum_{i=1}^{5} I_i(\phi) \\ I_1(\phi) &= \frac{1}{\varepsilon}\int_\Lambda [\sqrt{\varepsilon}\,\phi - :\sin(\sqrt{\varepsilon}\,\phi):]\sinh(\psi) \\ I_2(\phi) &= -\frac{\mu^2}{\varepsilon}\int \sqrt{\varepsilon}\,\phi\Delta\Delta\psi \\ I_3(\phi) &= \sum_{n=2}^{\infty} \operatorname{Im}[K_n(\phi + i\varepsilon^{-1/2}\psi) - K_n^0(\phi)] \\ I_4(\phi) &= \frac{1}{2\varepsilon}\sum_\gamma \int_\Lambda dx\ :\sin(\sqrt{\varepsilon}\,\phi):\ \gamma e^{-\gamma\psi}[e^{-\varepsilon\Delta_\gamma(x,y)} - 1] \\ I_5(\phi) &= \frac{1}{\varepsilon}\int_{\Lambda^c} \sqrt{\varepsilon}\,\phi\sinh(\psi) \end{aligned} \tag{5.26}$$

We will show

$$\langle |I_i(\phi)| \rangle = o(1) \qquad \text{for} \quad i = 1, 2, 3, \text{and } 4 \tag{5.27}$$

and

$$\langle |I_5(\phi)| \rangle = o(\Lambda) \tag{5.28}$$

First we show that this will complete our proof of Eq. (5.9). We have

$$|\langle e^{R(\phi)+iI(\phi)}\rangle - 1| \leqslant |\langle e^{R(\phi)}(e^{iI(\phi)} - 1)\rangle| + |\langle e^{R(\phi)} - 1\rangle| \tag{5.29}$$

Using $|e^{ix} - 1| \leqslant |x|$, (5.29) is

$$\leqslant \langle e^{R(\phi)}\,|I(\phi)|\rangle + \langle |e^{R(\phi)} - 1|\rangle$$

Our bound on $R(\phi)$, Eq. (5.20), is uniform in ϕ. So together with Eqs. (5.27) and (5.28) it implies Eq. (5.9).

To prove Eq. (5.27) for $i=1$ we use

$$\langle |I_1(\phi)| \rangle \leqslant \langle I_1(\phi)^2 \rangle^{1/2}$$

Now

$$\langle I_1(\phi)^2 \rangle = \frac{1}{\varepsilon^2} \int_\Lambda dx \int_\Lambda dy \sinh[\psi(x)] \sinh[\psi(y)] \\ \times \langle \{\sqrt{\varepsilon}\, \phi(x) - :\sin[\sqrt{\varepsilon}\, \phi(x)]:\} \{\sqrt{\varepsilon}\, \phi(y) - :\sin[\sqrt{\varepsilon}\, \phi(y)]:\} \rangle \tag{5.30}$$

By Lemma A.1 of Appendix A this is

$$= \frac{1}{\varepsilon^2} \int_\Lambda dx \int_\Lambda dy\, |\sinh[\psi(x)] \sinh[\psi(y)]|\, \sigma(\varepsilon^2) \left(1 + \frac{1}{|x-y|}\right)^{3/2}$$

Since $\sinh \psi \in L^1 \cap L^\infty$ [see Eq. (3.10)],

$$\int dx \int dy\, |\sinh[\psi(x)] \sinh[\psi(y)]| \left(1 + \frac{1}{|x-y|}\right)^{3/2} < \infty \tag{5.31}$$

Equation (5.27) for $i=1$ now follows.

For $i=2$ we use Cauchy–Schwartz as follows:

$$\langle |I_2(\phi)| \rangle \leqslant \frac{\mu^2}{\varepsilon} \int dx\, |\Delta\Delta\psi(x)|\, \langle \varepsilon\phi^2(x) \rangle^{1/2}$$

By Theorem 5.2 of [5],

$$\langle \varepsilon\phi^2(x) \rangle \leqslant \frac{\varepsilon}{4\pi\mu} \tag{5.32}$$

By our definition of μ, both μ^2/ε and $\varepsilon/\mu \to 0$ as $\varepsilon \to 0$. By (3.11), $\Delta\Delta\psi \in L^1$. So Eq. (5.27) is proven for $i=2$.

$I_3(\phi)$ and $I_4(\phi)$ are bounded in exactly the same way that $R_1(\phi)$ and $R_2(\phi)$ were in the previous step. For $i=5$ we have

$$\langle |I_5(\phi)| \rangle \leqslant \frac{1}{\varepsilon} \int_{\Lambda^c} dx\, |\sinh[\psi(x)]|\langle \varepsilon\phi^2(x) \rangle^{1/2} \\ \leqslant \frac{1}{\varepsilon} \left(\frac{\varepsilon}{4\pi\mu}\right)^{1/2} \int_{\Lambda^c} dx\, |\sinh[\psi(x)]| \tag{5.33}$$

by Eq. (5.32). Since $\sinh \psi \in L^1$,

$$\int_{\Lambda^c} dx\, |\sinh[\psi(x)]| = o(\Lambda)$$

So Eq. (5.28) is proven.

Step 6 [Proof of Eq. (5.10)].

$$\left\langle \prod_{j=1}^{m} :\exp[i\sqrt{\varepsilon}\, \delta_j \phi(y_j)]: \exp[R(\phi) + iI(\phi) + S'(\phi)] \right\rangle - 1$$
$$= T_1 + T_2 + T_3$$

where

$$\begin{aligned}
T_1 &= \left\langle \prod_{j=1}^{m} :\exp[i\sqrt{\varepsilon}\, \delta_j \phi(y_j)]: e^{R(\phi)+iI(\phi)}[e^{S'(\phi)} - 1] \right\rangle \\
T_2 &= \left\langle \prod_{j=1}^{m} :\exp[i\sqrt{\varepsilon}\, \delta_j \phi(y_j)]: [e^{R(\phi)+iI(\phi)} - 1] \right\rangle \\
T_3 &= \left\langle \prod_{j=1}^{m} :\exp[i\sqrt{\varepsilon}\, \delta_j \phi(y_j)]: - 1 \right\rangle
\end{aligned} \tag{5.34}$$

Equation (5.20) implies

$$|e^{R(\phi)+iI(\phi)}| = \mathcal{O}(1)$$

And

$$\left| \prod_{j=1}^{m} :\exp[i\sqrt{\varepsilon}\, \delta_j \phi(y_j)]: \right| \leqslant \exp\left(m \frac{\varepsilon}{8\pi\mu} \right) = \mathcal{O}(1) \tag{5.35}$$

So to show T_1 is $o(1)$ it suffices to show

$$|e^{S'(\phi)} - 1| = o(1) \tag{5.36}$$

$S'(\phi)$ is a difference of two Mayer series [Eq. (5.3)], so we can apply Lemma A.1 of [5]. We do this with

$$\begin{aligned}
V_1(x, \gamma) &= A_Y(x, \gamma) - \frac{i}{\sqrt{\varepsilon}} \gamma\phi(x) + \frac{1}{\varepsilon} \gamma\psi(x) \\
\bar{V}_1(x, \gamma) &= \bar{A}_S(x, \gamma) - A_Y(x, \gamma) \\
&= \sum_{j=1}^{m} V_S(x, \gamma; y_j, \delta_j)
\end{aligned} \tag{5.37}$$

The stability bound that V_ε is assumed to satisfy, Eq. (6.2) of [5], implies $V_S(x, \gamma; y_j, \delta_j) \geqslant -2B(\varepsilon)$. So by hypothesis (H3) $\varepsilon \| \bar{V}_1 \|_- \leqslant 2mB$. Hypotheses (H1) and (H2) and the choice of μ imply

$$\sum_\gamma \int dx \, |\exp[-\varepsilon \bar{V}_1(x, \gamma)] - 1| = \mathcal{O}(\varepsilon^2)$$

So Lemma A.1 of [5] says

$$|S'(\phi)| \leqslant \mathcal{O}(\varepsilon) \sum_{n=1}^{\infty} (r')^{n-1}$$

Since $r = \mathcal{O}(\varepsilon)$, $r' = \mathcal{O}(\varepsilon)$ and so

$$|S'(\phi)| = \mathcal{O}(\varepsilon) \tag{5.38}$$

which implies (5.36).

Using (5.35), the results of the previous two steps imply $T_2 = \mathcal{O}(1) + \mathcal{O}(\Lambda)$.

We have

$$\begin{aligned} T_3 = & \left\langle \exp\left[i\sqrt{\varepsilon} \sum_{j=1}^{m} \delta_j \phi(y_j) \right] \left[\exp\left(m \frac{\varepsilon}{8\pi\mu} \right) - 1 \right] \right\rangle \\ & + \left\langle \exp\left[i\sqrt{\varepsilon} \sum_{j=1}^{m} \delta_j \phi(y_j) \right] - 1 \right\rangle \end{aligned} \tag{5.39}$$

The first term is $\mathcal{O}(1)$ since $\varepsilon/\mu \to 0$ as $\varepsilon \to 0$. The absolute value of the second term is

$$\leqslant \left\langle \left| \sqrt{\varepsilon} \sum_{j=1}^{m} \delta_j \phi(y_j) \right| \right\rangle$$

By the Cauchy–Schwartz inequality and Eq. (5.32), this is

$$\leqslant m \left(\frac{\varepsilon}{4\pi\mu} \right)^{1/2}$$

which is $\mathcal{O}(1)$. Hence T_3 is $\mathcal{O}(1)$, which completes the proof of Eq. (5.10). ∎

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APPENDIX A

In this appendix we complete the proof in Section 5 by proving the bound used on Eq. (5.30). This bound is as follows.

Lemma A.1. Let $\langle\ \rangle$ be defined as in Section 5 [see Eqs. (5.7) and (5.5)]. Let μ depend on ε as in Eq. (5.11). Then

$$\langle\{\sqrt{\varepsilon}\,\phi(y) - {:}\sin[\sqrt{\varepsilon}\,\phi(y)]{:}\}\{\sqrt{\varepsilon}\,\phi(z) - {:}\sin[\sqrt{\varepsilon}\,\phi(z)]{:}\}\rangle$$

$$= o(\varepsilon^2)\left(1 + \frac{1}{|y-z|}\right)^{3/2} \tag{A.1}$$

Discussion. It is not hard to use Theorem 5.2 of [5] to show the expectation in Eq. (A.1) is $\mathcal{O}((\varepsilon/\mu)^3)$, which is $\mathcal{O}(e^{3/2-3\delta})$ by the definition of μ, Eq. (5.11). To show that the expectation is $o(\varepsilon^2)$ requires more than a simple application of Theorem 5.2 of [5].

The troublesome factor of $1/\mu$ in the above arises from the short-range behavior of the potential V_L. Short-range problems are better handled with the Mayer expansion than the sine-Gordon transformation. To implement this philosophy we write the Coulomb potential as

$$\frac{1}{4\pi|x|} = \mathscr{V}_{LL}(x) + \mathscr{V}_{LS}(x) + \mathscr{V}_{Y}(x)$$

with

$$\mathscr{V}_{LL}(x) = \frac{1 - \exp(-|x|/\gamma)}{4\pi|x|}$$

$$\mathscr{V}_{LS}(x) = \frac{\exp(-|x|/\gamma) - \exp(-|x|/\mu)}{4\pi|x|} \tag{A.2}$$

$$\mathscr{V}_{Y}(x) = \frac{\exp(-|x|/\mu)}{4\pi|x|}$$

and $\gamma > \mu$.

Until now we have been treating $\mathscr{V}_{LS}$ as part of the long-range part of the interaction and applying the sine-Gordon transformation to it. In the proof below we will treat $\mathscr{V}_{LS}$ as part of the short-range interaction. The parameter γ will be small but fixed. So the bounds on moments obtained from Theorem 5.2 of [5] will not have any ε dependence.

The reason we did not split up the Coulomb potential in this new way from the beginning is that for the translation $\phi \to \phi + i\varepsilon^{-1/2}\psi$ used in Section 5 to succeed, the part of the Coulomb interaction that is included in the Mayer expansion rather than the sine-Gordon transformation must go to zero sufficiently fast. With more work one can improve the bound (A.1). The bound stated is sufficient for our purposes.

Proof. Step 1 (Taylor's Theorem). For $\lambda, \delta \in [-1, 1]$ let

$$\begin{aligned} G(\lambda, \delta) &= \langle :e^{i\lambda\sqrt{\varepsilon}\phi(y)}::e^{i\delta\sqrt{\varepsilon}\phi(z)}:\rangle \\ H(\lambda, \delta) &= \tfrac{1}{2}[G(-\lambda, \delta) - G(\lambda, \delta)] \\ &= \langle :\sin[\lambda\sqrt{\varepsilon}\,\phi(y)]::\sin[\delta\sqrt{\varepsilon}\,\phi(z)]:\rangle \end{aligned} \tag{A.3}$$

since $\langle\ \rangle$ is even in ϕ. Then

$$\begin{aligned} &\langle\{\sqrt{\varepsilon}\,\phi(y) - :\sin[\sqrt{\varepsilon}\,\phi(y)]:\}\{\sqrt{\varepsilon}\,\phi(z) - :\sin[\sqrt{\varepsilon}\,\phi(z)]:\}\rangle \\ &\quad = H(1,1) + \frac{\partial^2 H}{\partial\lambda\,\partial\delta}(0,0) - \frac{\partial H}{\partial\lambda}(0,1) - \frac{\partial H}{\partial\delta}(1,0) \end{aligned}$$

Since H is odd in λ and δ, Taylor's theorem says this

$$= \int_0^1 \int_0^1 \frac{\partial^6 H}{\partial\lambda^3\,\partial\delta^3}(\lambda, \delta)\,\frac{(1-\lambda)^2}{2}\,\frac{(1-\delta)^2}{2}\,d\lambda\,d\delta \tag{A.4}$$

Let

$$Z(\lambda, \delta) = \int d\mu\ :e^{i\lambda\sqrt{\varepsilon}\phi(y)}::e^{i\delta\sqrt{\varepsilon}\phi(z)}:\ e^{W(\phi)} \tag{A.5}$$

with $W(\phi)$ defined as in Eq. (5.5). Then

$$G(\lambda, \delta) = \frac{Z(\lambda, \delta)}{Z(0, 0)}$$

To prove the lemma it suffices to show

$$\frac{\partial^6}{\partial\lambda^3\,\partial\delta^3}\,G(\lambda, \delta) = \sigma(\varepsilon^2)\left[1 + \frac{1}{|y - z|}\right]^{3/2} \tag{A.6}$$

Step 2 (Redoing sine-Gordon). Equation (5.5) for $W(\phi)$ shows that we can think of $\exp[W(\phi)]$ as the partition function of a system with four species. We label the species by $\alpha = 1, 2, 3, 4$. Species $\alpha = 1, 2$ come from the Mayer series in (5.5), and species $\alpha = 3, 4$ come from the term containing $:\cos(\sqrt{\varepsilon}\,\phi):$ in (5.5).

Species $\alpha = 3, 4$ have a "spatially dependent activity" $[\cosh(\psi) - 1]$. So we define a measure $d\nu(x, \alpha)$ on $\Lambda \times \{1, 2, 3, 4\}$ by

$$\int d\nu\,(x, \alpha)\, F(x, \alpha) = \sum_{\alpha=1,2} \int_\Lambda dx\, F(x, \alpha) + \sum_{\alpha=3,4} \int_\Lambda dx\, F(x, \alpha)\{\cosh[\psi(x)] - 1\}$$

All four species interact with the potential $-i\varepsilon^{-1/2}\phi(x)$. Let

$$\gamma(1) = \gamma(3) = +1$$
$$\gamma(2) = \gamma(4) = -1$$

So $\gamma(\alpha)$ is the charge of species α. Particles of species 1 and 2 interact by the two-body interaction V_S.

We undo the Mayer expansion and sine-Gordon transformation in our expression for $z(\lambda, \delta)$. Then we redo the transformation and expansion with $\mathscr{V}_{LS}$ included in the Mayer expansion rather than the sine-Gordon transformation. This is a straightforward calculation, so we omit the details.

Let $\mathscr{V}_{LL}$, $\mathscr{V}_{LS}$, and $\mathscr{V}_Y$ be defined as in Eq. (A.2). Let $d\hat{\mu}$, $\phi(x)$ be a Gaussian process with covariance $\mathscr{V}_{LL}(x-y)$. Then

$$Z(\lambda, \delta) = \int d\hat{\mu}\, Z(\lambda, \delta, \phi) \tag{A.7}$$

with

$$Z(\lambda, \delta, \phi) = \exp[-\varepsilon\lambda\, \delta \mathscr{V}_{LS}(y-z)] {:}\exp[i\lambda \sqrt{\varepsilon}\, \phi(y)]{:} \times {:}\exp[i\delta \sqrt{\varepsilon}\, \phi(z)]{:}\, \exp\left[\sum_{n=1}^{\infty} K_n(\lambda, \delta, \phi)\right] \tag{A.8}$$

In the Mayer series $\sum_{n=1}^\infty K_n(\lambda, \delta, \phi)$ the two-body potential is

$$\hat{V}_S(x, \alpha; x', \alpha') = \begin{cases} \gamma(\alpha)\, \gamma(\alpha')\, \mathscr{V}_{LS}(x-x') + V_S(x, \gamma(\alpha); x', \gamma(\alpha')) & \text{if } \alpha \text{ and } \alpha' \in \{1, 2\} \\ \gamma(\alpha)\, \gamma(\alpha')\, \mathscr{V}_{LS}(x-x') & \text{otherwise} \end{cases} \tag{A.9}$$

The one-body potential is

$$V_1(x, \alpha) = -i\varepsilon^{-1/2}\gamma(\alpha)\, \phi(x) + \lambda\gamma(\alpha)\, \mathscr{V}_{LS}(x-y) + \delta\gamma(\alpha)\, \mathscr{V}_{LS}(x-z) \tag{A.10}$$

Throughout this appendix K_n will refer to $K_n(\lambda, \delta, \phi)$, not to the Mayer series that was defined in Section 4. The Brydges and Federbush [2] form of the Mayer series is

$$K_n(\lambda, \delta, \phi) = \frac{(-\varepsilon)^{n-1}}{n} \frac{1}{(2\hat{\varepsilon})^n} \sum_\eta \int d\sigma\, f(\eta, \sigma) \int^n \hat{V}_S(\eta)$$
$$\times \exp[-\varepsilon \hat{V}_S^0(\sigma)]\, U(\sigma) \exp[i\sqrt{\varepsilon}\, \Phi - \varepsilon(\lambda \mathscr{V}_y + \delta \mathscr{V}_z)] \qquad \text{(A.11)}$$

with

$$\int^n = \prod_{i=1}^n \int dv\,(x_i, \alpha_i) \qquad \text{(A.12)}$$

$$\hat{\varepsilon} = \exp\left[-\frac{\varepsilon}{8\pi\gamma}\right]\varepsilon \qquad \text{(A.13)}$$

$$\Phi = \sum_i \gamma(\alpha_i)\, \phi(x_i)$$

$$\mathscr{V}_y = \sum_i \gamma(\alpha_i)\, \mathscr{V}_{LS}(x_i - y)$$

and $\mathscr{V}_z$ defined similarly. The rest of the notation in Eq. (A.11) is defined as in Eq. (A.2) of [5], except that wherever V_Y appears in [5] we have $V_Y + V_{LS}$. Rather than repeat the definitions in [5], we give a verbal description of various terms. $\sum_\eta$ is a sum over tree graphs. $\int d\sigma$ is an integral over the interpolation parameters $s_1, s_2, \ldots, s_{n-1}$. $f(\eta, \sigma)$ is a product of various s_i. $\hat{V}_S(\eta)$ is a product over the $n-1$ bonds in η of the two-body interactions associated with each bond. The term $\exp[-\varepsilon \hat{V}_S^0(\sigma)]\, U(\sigma)$ is $\exp[-\varepsilon \sum_{i<j} \hat{V}_S(i, j)]$ with the interpolation parameters s_i inserted in the appropriate manner. $U(\sigma)$ contains the repulsive part of $\hat{V}_S$, while $\exp[-\varepsilon \hat{V}_S^0(\sigma)]$ contains the rest of $\hat{V}_S$.

Step 3 (The Strategy). Let

$$F_0(\lambda, \delta, \phi) = -\varepsilon\lambda\, \delta \mathscr{V}_{LS}(y - z) + \lambda^2 \frac{\varepsilon}{8\pi\gamma} + i\lambda\sqrt{\varepsilon}\, \phi(y)$$
$$+ \delta^2 \frac{\varepsilon}{8\pi\gamma} + i\delta\sqrt{\varepsilon}\, \phi(z) \qquad \text{(A.14)}$$

$$F(\lambda, \delta, \phi) = F_0(\lambda, \delta, \phi) + \sum_{n=1}^\infty K_n(\lambda, \delta, \phi) \qquad \text{(A.15)}$$

So

$$Z(\lambda, \delta, \phi) = \exp[F(\lambda, \delta, \phi)]$$

Define $\langle\ \rangle'$ by

$$\langle I(\phi)\rangle' = Z(0,0)^{-1} \int d\hat{\mu}\, I(\phi) \exp[F(0,0,\phi)] \tag{A.16}$$

Then

$$G(\lambda, \delta) = \langle \exp[F(\lambda, \delta, \phi) - F(0,0,\phi)]\rangle'$$

So $\partial^6 G(\lambda, \delta)/\partial^3\lambda\, \partial^3\delta$ is a sum of terms of the form

$$\left\langle \prod_i [\partial^{\alpha_i} F(\lambda, \delta, \phi)] \exp[F(\lambda, \delta, \phi) - F(0,0,\phi)] \right\rangle' \tag{A.17}$$

Each α_i is a pair of nonnegative integers indicating the number of λ and δ derivatives, respectively. So $\sum_i \alpha_i = (3, 3)$.

We will show that each $\partial^\alpha F$ contributes $\mathcal{O}(\varepsilon^{(1/3+\delta')|\alpha|})$ to our bound on (A.17) with $\delta' > 0$. $|\alpha|$ is the sum of the two components of α. Each $\partial^\alpha F$ can also contribute a factor of $(1 + 1/|y-z|)^{|\alpha|/4}$ to our bound on (A.17). Since $\sum_i |\alpha_i| = 6$, we end up with the bound $o(\varepsilon^2)[1 + 1/|y-z|]^{3/2}$.

Step 4 [Handling $\phi(x)$'s]. Derivatives with respect to λ or δ can introduce factors of $\phi(x)$ in the $\langle\ \rangle'$ in (A.17). We bound these as follows. In step 8 we will show

$$|\exp[F(\lambda, \delta, \phi) - F(0,0,\phi)]| = \mathcal{O}(1) \tag{A.18}$$

A little thought shows $F(0,0,\phi)$ is real and $\langle\ \rangle'$ is a sine-Gordon probability measure. Using Hölder's inequality and Theorem 5.2 of [5]

$$\left| \left\langle \prod_{i=1}^m \phi(x_i) \exp[F(\lambda, \delta, \phi) - F(0,0,\phi)] \right\rangle' \right| \leqslant \mathcal{O}(1)[\mathscr{V}_{LL}(0)]^{m/2} \tag{A.19}$$

Since $\mathscr{V}_{LL}(0) = 1/4\pi\gamma$ and γ is fixed, we see that a factor of $\prod_{i=1}^m \phi(x_i)$ in (A.17) simply contributes a constant to our bound. This constant depends on m, but m will be at most 12.

Step 5 (Derivatives of F_0). From

$$\mathscr{V}_{LS}(0) = \frac{1}{4\pi\mu} - \frac{1}{4\pi\gamma} \leqslant \frac{1}{4\pi\mu}$$

and the definition of μ, Eq. (5.11), we have

$$\varepsilon \|\mathscr{V}_{LS}\|_\infty \leqslant \frac{\varepsilon}{\varepsilon^{1/2+\delta}} = \varepsilon^{1/2-\delta} \tag{A.20}$$

Since $\delta < 1/6$, $1/2-\delta > 1/3$. From Eq. (A.2) we also have the bound

$$|\mathscr{V}_{LS}(y-z)| \leqslant \frac{1}{4\pi |y-z|} \tag{A.21}$$

$\partial^\alpha F_0$ is not zero only for $|\alpha| = 1$ or 2. Equation (A.20) and step 4 easily establish the bound described at the end of step 3 for all α except $\alpha = (1, 1)$. $\partial^{(1,1)} F_0 = -\varepsilon \mathscr{V}_{LS}(y-z)$. We bound this case using both Eqs. (A.20) and (A.21).

$$\begin{aligned} |\varepsilon \mathscr{V}_{LS}(y-z)| &= |\varepsilon \mathscr{V}_{LS}(y-z)|^{1/2} \, |\varepsilon \mathscr{V}_{LS}(y-z)|^{1/2} \\ &\leqslant \varepsilon^{1/4-\delta/2} \varepsilon^{1/2} \left[\frac{1}{4\pi |y-z|} \right]^{1/2} \end{aligned}$$

Since $\delta < 1/6$, $1/4 - \delta/2 + 1/2 > 2/3 = |\alpha|/3$. We note that this is the only place where ∂^α can contribute powers of $1/|y-z|$.

Step 6 (Derivatives of $\sum_n K_n$). The usual techniques show

$$|K_n(\lambda, \delta, \phi)| \leqslant \mathscr{O}(\varepsilon^{-1}) \, |\Lambda| \, r^{n-1} \tag{A.22}$$

with $r = \mathscr{O}(1)\, \rho$ and

$$\rho = 2\mu^2 + c_1 \varepsilon^{1+\delta_1} + c_2 \varepsilon^{1+\delta_2} + 4\gamma^2 \tag{A.23}$$

(See the beginning of the proof in Appendix B for more details.) We can pick γ small but fixed so that for sufficiently small ε, $r \leqslant 1/2$. So the Mayer series in (A.8) converges.

Our bound on $\partial^\alpha K_n$ must be better than this bound on K_n in three ways. First, we must gain an extra factor of ε to cancel the $\mathscr{O}(\varepsilon^{-1})$ in (A.22). Second, we must gain a factor of $\mathscr{O}(\varepsilon^{1/3+\delta'})$ for each derivative in ∂^α. Third, our bound on $\partial^\alpha K_n$ must be uniform in Λ. It cannot contain a factor of $|\Lambda|$ like (A.22).

To gain the needed extra factor of ε we split up K_n.

$$K_n = \sum_{i=1}^{6} K_n^i$$

$$K_n^i = \frac{-1}{n\varepsilon} \left(\frac{-\varepsilon}{2\hat{\varepsilon}} \right)^n \sum_\eta \int d\sigma \, f(\eta, \sigma) \int^n \hat{V}_S(\eta) \, U(\sigma) \, I_i \tag{A.24}$$

$$I_1 = \{\exp[-\varepsilon \hat{V}_S^0(\sigma)] - 1\} \exp[i\sqrt{\varepsilon}\ \Phi] \exp[-\varepsilon\lambda \mathscr{Y}_y - \varepsilon\delta \mathscr{Y}_z]$$
$$I_2 = \{\exp[i\sqrt{\varepsilon}\ \Phi] - 1 - i\sqrt{\varepsilon}\ \Phi\} \exp[-\varepsilon\lambda \mathscr{Y}_y - \varepsilon\delta \mathscr{Y}_z]$$
$$I_3 = \exp[-\varepsilon\lambda \mathscr{Y}_y - \varepsilon\delta \mathscr{Y}_z] + [\varepsilon\lambda \mathscr{Y}_y + \varepsilon\delta \mathscr{Y}_z] - 1$$
$$I_4 = i\sqrt{\varepsilon}\ \Phi\{\exp[-\varepsilon\lambda \mathscr{Y}_y - \varepsilon\delta \mathscr{Y}_z] - 1\}$$
$$I_5 = -\varepsilon\lambda \mathscr{Y}_y - \varepsilon\delta \mathscr{Y}_z$$
$$I_6 = 1 + i\sqrt{\varepsilon}\ \Phi \tag{A.25}$$

K_n^6 is independent of λ and δ, so we can forget about it. We bound $\partial^\alpha K_n^5$ in the next step. For K_n^1, K_n^2, K_n^3, and K_n^4 we gain the needed extra factor of ε as follows.

$$\exp[-\varepsilon \hat{V}_S^0(\sigma)] - 1 = -\varepsilon \hat{V}_S^0(\sigma) \int_0^1 dt \exp[-t\varepsilon \hat{V}_S^0(\sigma)] \tag{A.26}$$

$$|\exp[i\sqrt{\varepsilon}\ \Phi] - 1 - i\sqrt{\varepsilon}\ \Phi| \leqslant \frac{\varepsilon}{2} \Phi^2 \tag{A.27}$$

$$\begin{aligned} &\exp[-\varepsilon\lambda \mathscr{Y}_y - \varepsilon\delta \mathscr{Y}_z] + [\varepsilon\lambda \mathscr{Y}_y + \varepsilon\delta \mathscr{Y}_z] - 1 \\ &\quad = \varepsilon^2(\lambda \mathscr{Y}_y + \delta \mathscr{Y}_z)^2 \int_0^1 (1-t) \exp[-t\varepsilon(\lambda \mathscr{Y}_y + \delta \mathscr{Y}_z)]\, dt \end{aligned} \tag{A.28}$$

$$\begin{aligned} &\exp[-\varepsilon\lambda \mathscr{Y}_y - \varepsilon\delta \mathscr{Y}_z] - 1 \\ &\quad = -\varepsilon(\lambda \mathscr{Y}_y + \delta \mathscr{Y}_z) \int_0^1 \exp[-t\varepsilon(\lambda \mathscr{Y}_y + \delta \mathscr{Y}_z)]\, dt \end{aligned} \tag{A.29}$$

Note that in addition to the needed extra factor of ε we gain an additional ε in K_n^3, while K_n^4 contains an additional $\sqrt{\varepsilon}$.

In K_n^1, K_n^2, K_n^3, and K_n^4 derivatives with respect to λ or δ can act on the λ or δ in the exponential and bring down factors of $\varepsilon \mathscr{Y}_y$ or $\varepsilon \mathscr{Y}_z$. We use Eq. (A.20) to extract $\mathscr{O}(\varepsilon^{1/3+\delta'})$ from these factors. In K_n^3 two of the derivatives in ∂^α can act on the $(\lambda \mathscr{Y}_y + \delta \mathscr{Y}_z)^2$ term. The extra factor of ε from Eq. (A.28) gives a factor of $\varepsilon^{1/2}$ for each such derivative. In K_n^4 one of the derivatives in ∂^α can act on the factor of $(\lambda \mathscr{Y}_y + \delta \mathscr{Y}_z)$ outside the dt integral. The extra factor of $\sqrt{\varepsilon}$ is associated with this derivative.

Each of Φ, $\mathscr{Y}_y$ and $\mathscr{Y}_z$ contain n terms. So our bounds for $\partial^\alpha K_n^i$ will contain polynomials in n. This is not a problem since

$$\sum_{n=1}^{\infty} P(n)\, r^{n-1} < \infty$$

for any polynomial $P(n)$ and $r < 1$.

In K_n^1 and K_n^2 we avoid a factor of $|\Lambda|$ in our bound on $\partial^\alpha K_n^i$ by keeping one of the factors of $\varepsilon\mathscr{V}_y$ or $\varepsilon\mathscr{V}_z$ brought down by ∂^α rather than using Eq. (A.20). Then we use Eqs. (B.1) and (B.2) of Lemma B.1 to bound the $\int^n$ integral. We bound the $\int^n$ integral in K_n^3 and K_n^4 using Eqs. (B.3) and (B.2) of Lemma B.1. The factors of $(\lambda\mathscr{V}_y+\delta\mathscr{V}_z)^2$ and $(\lambda\mathscr{V}_y+\delta\mathscr{V}_z)$ in (A.28) and (A.29) prevent a factor of $|\Lambda|$ in these bounds.

Step 7 (Bounding $\partial^\alpha K_n^5$). Recall that $\hat{V}_S$ is V_S+V_{LS}. So we can expand $\hat{V}_S(\eta)$ as a sum of 2^{n-1} terms. We separate out the one term which has the potential V_{LS} associated with every bond in η. It is

$$V^1(\eta)=\prod_{i=1}^{n-1} V_{LS}(i+1,\eta(i)) \tag{A.30}$$

We let the sum of the other $2^{n-1}-1$ terms be $V^2(\eta)$, so

$$\hat{V}_S(\eta)=V^1(\eta)+V^2(\eta) \tag{A.31}$$

We will show that the $V^1(\eta)$ part of K_n^5 is 0. The $V^2(\eta)$ part of K_n^5 will $\to 0$ as $\varepsilon\to 0$ since each term in $V^2(\eta)$ contains at least one factor of $V_S(i+1,\eta(i))$, which $\to 0$ as $\varepsilon\to 0$.

Consider

$$V^1(\eta)\,U(\sigma)(\lambda\mathscr{V}_y+\delta\mathscr{V}_z) \tag{A.32}$$

We claim that when summed over $\alpha_1,\ldots,\alpha_n$ it gives zero. (A.32) is a sum of $2n$ terms like

$$V^1(\eta)\,U(\sigma)\,\lambda\gamma(\alpha_i)\,\mathscr{V}_{LS}(x_i-y) \tag{A.33}$$

Every tree graph η has at least two vertices which have only one line hitting them. Let j be such a vertex with j not equal to i.

Consider the sum of (A.33) over α_j. The only factor in (A.33) which depends on α_j is $V^1(\eta)$. (One of the hypotheses on V_ε is that the repulsive part of V_ε is independent of the species of the particles. See p. 279 of [5]. So $U(\sigma)$ is independent of α_j). $V^1(\eta)$ contains exactly one factor of $\gamma(\alpha_j)$ since only one line hits vertex j. Hence

$$\sum_{\alpha_j} V^1(\eta)=0$$

This proves our claim.

Thus we can replace $\hat{V}_S(\eta)$ by $V^2(\eta)$ in K_n^5. $\partial^\alpha K_n^5$ is nonzero only for $|\alpha|=1$. I_5 already contains the needed extra factor of ε. Equation (B.4) of Lemma B.1 provides another factor of ε to go with the single derivative in ∂^α.

Step 8. We prove Eq. (A.18) from step 4. It is easy to check that

$$|\exp[F_0(\lambda, \delta, \phi)]| = \mathcal{O}(1)$$

So it suffices to show

$$\left| \sum_{n=1}^{\infty} [K_n(\lambda, \delta, \phi) - K_n(0, 0, \phi)] \right| = \mathcal{O}(1)$$

We use

$$K_n(\lambda, \delta, \phi) - K_n(0, 0, \phi) = \int_0^1 \frac{d}{dt} K_n(t\lambda, t\delta, \phi)\, dt$$

The d/dt brings down a factor of $-\varepsilon(\lambda \mathscr{V}_y + \delta \mathscr{V}_z)$. This gives the needed extra factor of ε. The $\int^n$ integral is bounded using Eq. (B.2) of Lemma B.1. ∎

APPENDIX B

We prove the bounds on Mayer graphs that were used in the proof in Appendix A.

Lemma B.1. Let $\int^n$, $\hat{V}_S(\eta)$ and $U(\sigma)$ be defined as in Appendix A. [See Eq. (A.12) and step 2 of the proof in Appendix A.] Then for $x, x' \in \mathbb{R}^3$ and $1 \leqslant i, j, k \leqslant n$ we have the following estimates:

$$\int^n |\hat{V}_S(\eta)\, U(\sigma)\, \hat{V}_S^0(j, k)\, \mathscr{V}_{LS}(x_i - x)| = \mathcal{O}(1)\, \rho^{n-2} \tag{B.1}$$

$$\int^n |\hat{V}_S(\eta)\, U(\sigma)\, \mathscr{V}_{LS}(x_i - x)| = \mathcal{O}(1)\, \rho^{n-1} \tag{B.2}$$

$$\int^n |\hat{V}_S(\eta)\, U(\sigma)\, \mathscr{V}_{LS}(x_i - x)\, \mathscr{V}_{LS}(x_j - x')| = \mathcal{O}(1)\, \rho^{n-1} \tag{B.3}$$

$$\int^n |V^2(\eta)\, U(\sigma)\, \mathscr{V}_{LS}(x_i - x)| = o(\varepsilon)\, \rho^{n-2} \tag{B.4}$$

$V^2(\eta)$ is defined in step 7 of Appendix A. ρ is given by Eq. (A.23).

Proof. We begin by reviewing the usual technique for bounding

$$\int^n |\hat{V}_S(\eta)\, U(\sigma)|$$

We think of η as a tree graph, e.g., see Fig. 1. Starting with the vertices which have only one line attached to them, we use hypotheses (H1) and (H2) to bound the integrations over the associated (x_i, α_i) by ρ.

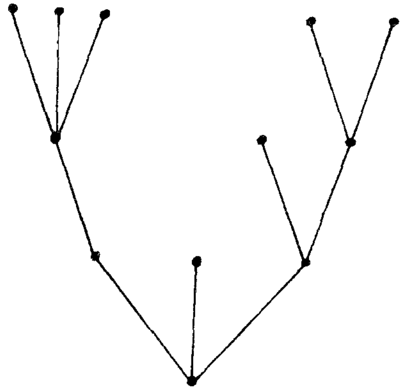

Fig. 1. A typical tree graph η.

We repeat this procedure until all of the lines in the tree have been bounded. There are $n-1$ lines, so we end up with a bound of ρ^{n-1}. The last integration gives

$$\int dv\,(x_i, \alpha_i) = 4\,|\Lambda|$$

Note that you can take any vertex to be the base of the tree, and so bound the corresponding integration last.

To prove (B.2) we take vertex i as the base of the tree graph. We proceed as above, ending up with

$$\int dv\,(x_i, \alpha_i)\,|\mathscr{V}_{LS}(x_i - x)|$$

This integral is $\mathscr{O}(1)$ by explicit computation.

To prove (B.3) we use the Cauchy Schwartz inequality to reduce it to the case of $i = j$, $x = x'$. We proceed as before, ending up with

$$\int dv\,(x_i, \alpha_i)\,|\mathscr{V}_{LS}(x_i - x)|^2$$

which is $\mathscr{O}(1)$.

We prove (B.4) in the same manner as (B.2). However, at least one of the lines in η corresponds to just V_S rather than $\hat{V}_S = V_S + V_{LS}$. By (H1) and (H2) such lines will contribute $\sigma(\varepsilon)$ instead of just ρ.

The proof of (B.1) is slightly more complicated. We represent $\hat{V}_S^0$ by a dashed line. So $\hat{V}_S(\eta)\,\hat{V}_S^0(j, k)$ is the Mayer graph η with a dashed line from vertex j to vertex k added. We take vertex i as the base of the tree. We bound integrations over vertices with only one line as before. However, the graph now has a loop in it. So we end up with the graph shown in Fig. 2. Of

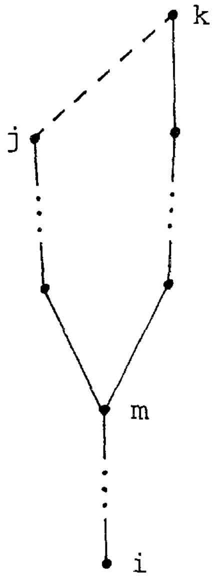

Fig. 2. The $\hat{v}_S^0(j,k)$ term, represented by a ---, introduces a loop in the graph. The •——···——• stands for an unspecified number of repetitions of •——•.

course, one of j or k can equal m, but not both. Assume j is not equal to m. We break the loop by applying the Cauchy–Schwartz inequality to the integration over vertex j. The resulting integrals are bounded by hypothesis (H4) and the computation

$$\int dx\,|\mathscr{V}_Y(x)+\mathscr{V}_{LS}(x)|^2=\mathscr{O}(1)$$

We fail to gain a factor of ρ from the line hitting vertex j when we do this. Having broken the loop we continue as before. ∎

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